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On the mechanics of the bow and arrow

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SUMMARY

Some aspects of the dynamics of the bow and arrow have been considered. The governing equations are derived by means of Hamilton's principle. The resulting non-linear initial-boundary-value problem is solved numerically by use of a finite-difference method. The influence of the characteristic quantities on the performance of a bow is discussed.

1. Introduction

This paper deals with the interior ballistic of the bow and arrow, hence with the phenomena which happen between the moment of release of the arrow and the moment that the arrow leaves the string. This subject is amply investigated experimentally by Hickman and Klopsteg [1]. Hickman used also a mathematical model. In order to be able to get numerical results without the help of a computer his model had rather severe simplifications. Because of these simplifications only bows with specific features could be dealt with. We hope that this article will add to the understanding of the action of rather general types of bows, by giving more accurate and detailed numerical results.

We are concerned with bows of which the flexible parts (limbs) move in a flat plane, and which are symmetric with respect to the line of aim. The arrow will pass through the midpoint of the bow, as in the case of a 'centre-shot bow'. We assume that the bow is clamped at its midpoint by the bow hand. The bows are allowed to possess a mild 'recurve' of 'reflex'. This means that the limbs of the bow in unstrung situation are allowed to be curved away from the archer, however, not too strongly.

We will consider the bow as a slender inextensible beam. The dynamical boundary conditions at the tips of the elastic limbs follow from the connection of the tips, by means of a string, to the end of the arrow. The initial deformation of the bow is given by its shape in the fully drawn position, the initial velocities are zero. Also in our theory some assumptions have been made. Most of these result from the use of the Euler-Bernoulli equation for the elastic line which represents the bow. Further, the mass of the string is taken to be zero, the string is assumed to be inextensible and the arrow is taken to be rigid. Neither internal or external damping nor hysteresis are taken into account.

Non-linear vibrations of beams have been studied by many authors. Most of them are concerned with periodic motions. Woodall [2] obtains the governing equations of motion by con-

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sidering a differential element of a beam. Wagner [3] and later Verma and Krishna Murthy [4] applied Hamilton's principle. However, in [3] and [4] the constraint which follows from the fact that the beam is assumed to be inextensible is not taken into account in the variational problem itself, but is used afterwards. This makes their equations differ from ours. In Section 2 Hamilton's principle is used and a physical meaning of the Lagrange multiplier connected to the inextensibility of the bow is given. This has been done by comparing our equations with those obtained by Woodall. In the static case such a method was already applied by Schmidt and Da Deppo [5].

In Section 3 a finite-difference method to solve the equations of motion numerically is described. The results are compared with the results of a finite-element method.

The performance of a bow and arrow depends on a number of parameters, the length of the bow, the brace height or the length of the string, the draw, the mass of the arrow and the mass of concentrated masses at the tips (if any). It depends also on three functions, namely the distributions of bending stiffness and mass along the bow and the shape of the bow in its unstrung situation. In order to get insight into the influence of the afore-mentioned quantities, in Section 4 these quantities are changed systematically, starting from a bow described by Hickman [1], page 69. Besides the static quality coefficient, already introduced in [6], two dynamic quality coefficients are introduced. One is the efficiency and the other is related to the velocity of the arrow when it leaves the string, sometimes called the muzzle velocity. These three numbers cannot give by themselves a complete insight into what makes a bow a good one, for instance, with respect to target shooting, flight shooting or hunting. Also other subjects become important, such as smoothness of the recoil of the bow, its manageability, and so on. Whenever it is possible, our results are compared with experimental and theoretical results given in [1].

Although it belongs clearly to the interior ballistics of a bow and arrow, we will not discuss in this paper the interesting 'archer's paradox'. This is the phenomenon that the elastic arrow, during the shooting period of a conventional non-centre-shot bow, carries out a vibrational motion. Because we only consider centre-shot bows, the assumption that the arrow be rigid with respect to bending is without loss of generality.

In the Appendix some attention is paid to the behaviour of the normal or longitudinal force in our model of the bow, at the moment the arrow is released. When concentrated masses at the tips are present, the normal force seems to have a jump at that moment. This jump disappears when in an approximate way extensibility of the bow is simulated.

2. Equations of motion

In this section the equations of motion of the bow and the dynamic boundary conditions are derived by means of Hamilton's principle. The equations of motion can also be obtained by applying the linear momentum and angular momentum balance of a differential element of the bow, as is done for instance by Woodall [2].

First we introduce the quantities which fix, with respect to our problem, the features of bow and arrow. The total length of the inextensible bow is denoted by $2\bar{L}$. The bow will be represented by an elastic line, along which we have a length coordinate $\bar{s}$, measured from the midpoint, hence $0 \leq |\bar{s}| \leq \bar{L}$. This elastic line is endowed with bending stiffness $\bar{W}(\bar{s})$ and mass per unit of length $\bar{V}(\bar{s})$. The rigid arrow has a mass $2\bar{m}_a$, where the factor 2 is inserted for convenience later on. In addition, there may be concentrated masses $\bar{m}_t$ at each of the tips, representing the mass of the horns used to fasten the string or artificially added masses. The bow is placed in a Cartesian coordinate system $(\bar{x}, \bar{y})$, the $\bar{x}$-axis coinciding with the line of aim and the origin 0 with the centre of the bow. Because the bow is symmetric with respect to the line of aim, only the upper half of the bow is dealt with in what follws. The unbraced situation (Figure 2.1(a)) is given by the functions $\bar{x} = \bar{x}_0(\bar{s})$ and $\bar{y} = \bar{y}_0(\bar{s})$ or by the angle $\theta_0(\bar{s})$ between the $\bar{y}$-axis and the tangent to the bow, reckoned positive in clockwise direction. Because $\bar{s}$ is the length parameter the functions $\bar{x}_0(\bar{s})$ and $\bar{y}_0(\bar{s})$ have to satisfy $\bar{x}_0'^2(\bar{s}) + \bar{y}_0'^2(\bar{s}) = 1$, where the prime indicates differentiation with respect to $\bar{s}$. $\bar{L}_0$ is the half length of the rigid part in the middle of the bow, called the 'grip', thus for $0 \leq \bar{s} \leq \bar{L}_0$ we have $\bar{W}(\bar{s}) = \infty$.

In Figure 2.1(b) the braced situation is depicted. The distance $|\overline{OH}|$ is the 'brace height' or 'fist mele'. The length of the inextensible string, used to brace the bow, is denoted by $2\bar{l}\,(\bar{l} < \bar{L})$. It is possible that, when recurve is present, the string lies along part of the bow in the braced situation. However, in this paper we assume the string to have contact with the bow only at the tips in all situations, static or dynamic. Hence, only bows without recurve or with a moderate recurve will be considered. In Figure 2.1(c) the bow is in fully drawn position. The geometry in this position is described by the functions $\bar{x} = \bar{x}_1(\bar{s})$ and $\bar{y} = \bar{y}_1(\bar{s})$ $(\bar{x}_1'^2(\bar{s}) + \bar{y}_1'^2(\bar{s}) = 1)$, or by the angle $\theta_1(\bar{s})$. The distance $|\overline{OD}|$ is called the 'draw' and the force $\bar{F}(|\overline{OD}|)$ is the 'weight'.

The following short notation of a specific bow and arrow combination will be used:

$$B(\bar{L}, \bar{L}_0, \bar{W}(\bar{s}), \bar{V}(\bar{s}), \theta_0(\bar{s}), \bar{m}_a, \bar{m}_t, |\overline{OH}| \text{ or } \bar{l};\ |\overline{OD}|, \bar{F}(|\overline{OD}|), \bar{m}_b), \tag{2.1}$$

where the brace height $|\overline{OH}|$ or half of the length $\bar{l}$ of the string can be given. Further $\bar{m}_b$ is half of the mass of the limbs, the flexible parts of the bow, so

$$\bar{m}_b = \int_{\bar{L}_0}^{\bar{L}} \bar{V}(\bar{s})\,\mathrm{d}\bar{s}. \tag{2.2}$$

The variables before the semicolon in (2.1) together with the draw $|\overline{OD}|$ determine completely the features of the bow, while the quantities behind the semicolon are used when we introduce dimensionless variables.

We now derive the equations of motion of bow and arrow. For simplicity we take $\bar{L}_0 = 0$; if

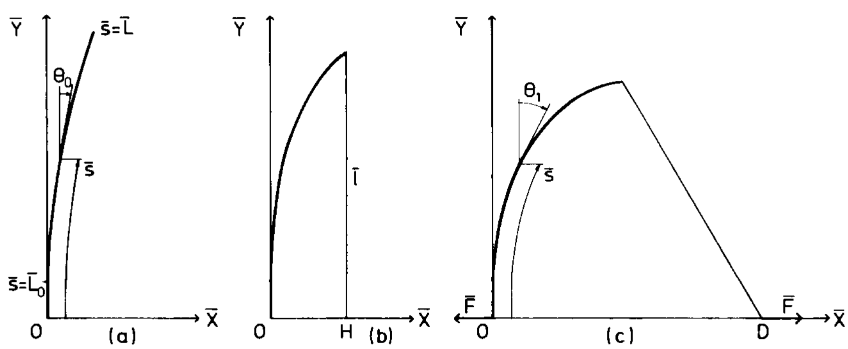

Figure 2.1. Three situations of a bow: (a) unbraced, (b) braced and (c) fully drawn.

this is not the case the obtained equations have to be changed in an obvious way. The Bernoulli-Euler equation (which is assumed to be valid) reads

$$\bar{M}(\bar{s},\bar{t}) = \bar{W}(\bar{s}) \quad \{\bar{x}'\bar{y}'' - \bar{y}'\bar{x}'' + \theta_0'\}, \quad 0 \le \bar{s} \le \bar{L}, \tag{2.3}$$

where $\bar{M}(\bar{s},\bar{t})$ is the resultant bending moment at a cross section (see Figure 2.2. for sign).
We recall that because the bow is symmetric with respect to the line of aim, we confine ourselves to its upper half, clamped at the origin 0. The potential energy $\bar{A}_p$ of the deformed upper half is its bending energy

$$\bar{A}_p = \tfrac{1}{2} \int_0^{\bar{L}} \frac{\bar{M}^2(\bar{s},\bar{t})}{\bar{W}(\bar{s})} \, d\bar{s}. \tag{2.4}$$

The kinetic energy $\bar{A}_k$ is the sum of the kinetic energy of the upper half of the bow, half the kinetic energy of the arrow and the kinetic energy of the concentrated mass at the tip. Then when a dot indicates differentiation with respect to time $\bar{t}$,

$$\bar{A}_k = \tfrac{1}{2} \int_0^{\bar{L}} \bar{V}(\bar{s})\,(\dot{\bar{x}}^2 + \dot{\bar{y}}^2)\, d\bar{s} + \tfrac{1}{2}\bar{m}_a \dot{\bar{b}}^2 + \tfrac{1}{2}\bar{m}_t \{\dot{\bar{x}}^2(\bar{L},\bar{t}) + \dot{\bar{y}}^2(\bar{L},\bar{t})\}, \tag{2.5}$$

where $\bar{b}$ is the $\bar{x}$-coordinate of the end of the arrow or the middle of the string, which can be expressed in the coordinates of the tip of the bow by

$$\bar{b}(\bar{t}) = \bar{x}(\bar{L},\bar{t}) + (\bar{l}^2 - \bar{y}^2(\bar{L},\bar{t}))^{\frac{1}{2}}, \tag{2.6}$$

because the string is assumed to be inextensible. The string is also assumed to be without mass, hence it contributes neither to the potential nor to the kinetic energy. Because the bow is inextensional we have the constraint

$$\bar{x}'^2 + \bar{y}'^2 = 1, \quad 0 \le \bar{s} \le \bar{L}. \tag{2.7}$$

We introduce

$$\bar{\Lambda} = \bar{A}_k - \bar{A}_p + \int_0^{\bar{L}} \bar{\lambda}(\bar{s},\bar{t})\,(\bar{x}'^2 + \bar{y}'^2 - 1)\, d\bar{s}, \quad 0 \le \bar{s} \le \bar{L}, \tag{2.8}$$

where $\bar{\lambda}(\bar{s},\bar{t})$ is an unknown Lagrangian multiplier to meet the constraint (2.7). Then by Hamilton's principle we have to find an extremum of

$$\int_{\bar{t}_0}^{\bar{t}_1} \bar{\Lambda}\, d\bar{t}, \tag{2.9}$$

hence

$$\delta \int_{\bar{t}_0}^{\bar{t}_1} \bar{\Lambda}\, d\bar{t} = 0, \tag{2.10}$$

for fixed initial time $\bar{t} = \bar{t}_0$ and fixed final time $\bar{t} = \bar{t}_1$.

Because the bow is clamped at the origin 0, we have for $\bar{s} = 0$ the geometric boundary conditions

$$\bar{x}(0,\bar{t}) = \bar{y}(0,\bar{t}) = 0, \quad \bar{y}'(0,\bar{t}) = \bar{y}'_0(0). \tag{2.11}$$

By standard methods of calculus of variations and using (2.11) we find the Euler equations as necessary conditions for the extremum of (2.9)

$$\bar{V}\ddot{\bar{x}} = (\bar{y}''\bar{M})' - 2(\bar{\lambda}\bar{x}')' + (\bar{y}'\bar{M})'', \tag{2.12}$$

and

$$\bar{V}\ddot{\bar{y}} = -(\bar{x}''\bar{M})' - 2(\bar{\lambda}\bar{y}')' - (\bar{x}'\bar{M})''. \tag{2.13}$$

Also the dynamic boundary conditions at $\bar{s} = \bar{L}$ follow from the variational procedure, they become

$$\bar{M}(\bar{L},\bar{t}) = 0, \tag{2.14}$$

$$\bar{m}_a\ddot{\bar{b}} + \bar{m}_t\ddot{\bar{x}}(\bar{L},\bar{t}) = 2\bar{\lambda}(\bar{L},\bar{t})\bar{x}'(\bar{L},\bar{t}) - \bar{y}'(\bar{L},\bar{t})\bar{M}'(\bar{L},\bar{t}), \tag{2.15}$$

$$\bar{m}_a\frac{\bar{y}(\bar{L},\bar{t})\ddot{\bar{b}}}{\bar{b}-\bar{x}(\bar{L},\bar{t})} - \bar{m}_t\ddot{\bar{y}}(\bar{L},\bar{t}) = -2\bar{\lambda}(\bar{L},\bar{t})\bar{y}'(\bar{L},\bar{t}) - \bar{x}'(\bar{L},\bar{t})\bar{M}'(\bar{L},\bar{t}). \tag{2.16}$$

The initial conditions which complete the formulation of the problem are

$$\bar{x}(\bar{s},0) = \bar{x}_1(\bar{s}),\ \bar{y}(\bar{s},0) = \bar{y}_1(\bar{s}),\ \dot{\bar{x}}(\bar{s},0) = \dot{\bar{y}}(\bar{s},0) = 0, \quad 0 \le \bar{s} \le \bar{L}. \tag{2.17}$$

Although it is not necessary for the computations, we look for a physical meaning of the function $\bar{\lambda}(\bar{s},\bar{t})$. In Figure 2.2 the resultant forces and moments acting on a differential element of the bow are shown. The momentum balance in the $\bar{x}$-and $\bar{y}$-direction gives

$$\bar{V}\ddot{\bar{x}} = (\bar{T}\bar{x}')' - (\bar{Q}\bar{y}')', \tag{2.18}$$

and

$$\bar{V}\ddot{\bar{y}} = (\bar{T}\bar{y}')' + (\bar{Q}\bar{x}')', \tag{2.19}$$

respectively, where $\bar{T}(\bar{s},\bar{t})$ is the normal force and $\bar{Q}(\bar{s},\bar{t})$ the shear force on a cross section (see Figure 2.2). If the rotatory inertia of the cross section of the bow is neglected, the moment balance of the element gives

$$\bar{M}'(\bar{s},\bar{t}) = -\bar{Q}(\bar{s},\bar{t}). \tag{2.20}$$

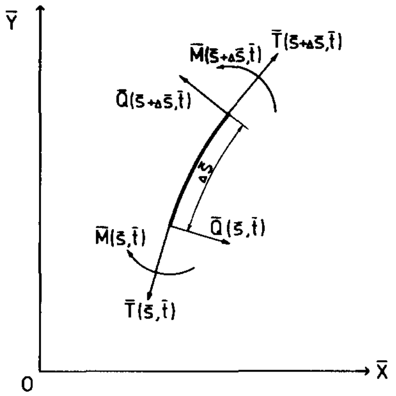

Figure 2.2. Forces and moments on a differential element of the bow.

Comparing equations (2.18) and (2.19), using (2.20) to replace $\bar{Q}$ by $\bar{M}'$, with (2.12) and (2.13), we find the physical meaning of $\bar{\lambda}$

$$\bar{\lambda}(\bar{s},\bar{t}) = -\tfrac{1}{2}\bar{T} + \frac{\bar{M}}{\bar{W}}(\bar{M} - \bar{W}\theta_0'), \quad 0 \le \bar{s} \le \bar{L}. \tag{2.21}$$

Substitution of (2.21) in the boundary conditions (2.15) and (2.16) yields

$$\bar{m}_a\ddot{\bar{b}} + \bar{m}_t\ddot{\bar{x}}(\bar{L},\bar{t}) = -\bar{T}(\bar{L},\bar{t})\bar{x}'(\bar{L},\bar{t}) - \bar{y}'(\bar{L},\bar{t})\bar{M}'(\bar{L},\bar{t}), \tag{2.22}$$

and

$$\bar{m}_a\frac{\bar{y}(\bar{L},\bar{t})}{\bar{b}(\bar{t})-\bar{x}(\bar{L},\bar{t})}\ddot{\bar{b}} - \bar{m}_t\ddot{\bar{y}}(\bar{L},\bar{t}) = \bar{T}(\bar{L},\bar{t})\bar{y}'(\bar{L},\bar{t}) - \bar{x}'(\bar{L},\bar{t})\bar{M}'(\bar{L},\bar{t}). \tag{2.23}$$

Equations (2.22) and (2.23) connect the deformation of the bow at $\bar{s} = \bar{L}$ to the force components in the $\bar{x}$- and $\bar{y}$-direction, exerted by the string and by the mass $\bar{m}_t$ at the tip.

The functions $\bar{x}_1(\bar{s})$ and $\bar{y}_1(\bar{s})$ occurring in the initial conditions (2.17) satisfy the equations of static equilibrium, with $\bar{b} = |\overline{OD}|$, obtained from (2.18), (2.19) and (2.20) by putting the left-hand sides of the first two mentioned equations equal to zero. The two relations (2.3) and (2.7) remain unchanged. Besides we have the boundary conditions (2.11), (2.14), (2.22) and (2.23), where in the latter two we have to replace the first term on the left-hand sides by $-\tfrac{1}{2}\bar{F}(|\overline{OD}|)$ and $-\tfrac{1}{2}\bar{y}_1(\bar{L})\ \bar{F}(|\overline{OD}|)/(\bar{b} - \bar{x}_1(\bar{L}))$, respectively. The weight of the bow $\bar{F}(|\overline{OD}|)$ is unknown and has to be determined in the course of the solution of these equations. In Equation (2.6) $\bar{b}$ has to be replaced by its known value $|\overline{OD}|$, the draw of the bow. The static deformations are discussed in [6].

The acceleration (or dynamic) force on the arrow, denoted by $\bar{E}$, is given by

$$\bar{E}(\bar{t}) = -2\bar{m}_a\ddot{\bar{b}}(\bar{t}). \tag{2.24}$$

and the recoil force $\bar{P}$, which is the force of the bow exerted on the bow hand (reckoned positive in the positive x-direction) by

$$\bar{P}(\bar{t}) = 2\{\bar{M}'(0,\bar{t})\,\bar{y}_0'(0) + \bar{T}(0,\bar{t})\,\bar{x}_0'(0)\}. \tag{2.25}$$

We introduce dimensionless quantities in the following way

$$(\bar{x},\bar{y},\bar{s},\bar{L},\bar{L}_0,\bar{l},|\overline{OH}|,\bar{b}) = (x,y,s,L,L_0,l,|OH|,b)\cdot|\overline{OD}|,$$

$$(\bar{T},\bar{F},\bar{E},\bar{P}) = (T,F,E,P)\cdot\bar{F}(|\overline{OD}|),$$

$$\bar{M} = M\cdot|\overline{OD}|\cdot\bar{F}(|\overline{OD}|),\ \bar{W} = W|\overline{OD}|^2\cdot\bar{F}(|\overline{OD}|),\ \bar{V} = V\cdot\bar{m}_b/|\overline{OD}|,$$

$$(\bar{m}_a,\bar{m}_t) = (m_a,m_t)\cdot\bar{m}_b,\ \bar{t} = t\cdot\{\bar{m}_b|\overline{OD}|/\bar{F}(|\overline{OD}|)\}^{\frac{1}{2}}, \tag{2.26}$$

where we used the a priori unknown weight $\bar{F}(|\overline{OD}|)$ of the bow to make the quantities dimensionless. In the following we will systematically label quantities with dimension by means of a bar '–', quantities without bar are dimensionless. Quantities, when they have dimensions, will be expressed, *unless stated otherwise*, by means of the following units: length in cm, force in kgforce, mass in kgmass and time in .03193 sec.

If the velocities $u(s,t) = \dot{x}(s,t)$ and $v(s,t) = \dot{y}(s,t)$ are introduced the system of six non-linear partial differential equations for the six functions x, y, u, v, M, T of two independent variables $s \in [L_0, L]$ and $t > 0$ assumes the form

$$V\dot{u} = (Tx')' + (M'y')', \tag{2.27}$$

$$V\dot{v} = (Ty')' - (M'x')', \tag{2.28}$$

$$\dot{x} = u, \tag{2.29}$$

$$\dot{y} = v, \tag{2.30}$$

$$x'^2 + y'^2 = 1, \tag{2.31}$$

$$M = W(x'y'' - y'x'' + \theta_0'). \tag{2.32}$$

The boundary conditions at $s = L_0$ become

$$x(L_0,t) = x_0(L_0),\ y(L_0,t) = y_0(L_0),\ y'(L_0,t) = y_0'(L_0), \tag{2.33}$$

and at $s = L$ $(t > 0)$,

$$M(L,t) = 0, \tag{2.34}$$

$$m_a\ddot{b} + m_t\ddot{x}(L,t) = -T(L,t)x'(L,t) - M'(L,t)y'(L,t), \tag{2.35}$$

$$m_a y(L,t)\ddot{b} - m_t\ddot{y}(L,t)(b(t) - x(L,t)) = (T(L,t)y'(L,t) - M'(L,t)x'(L,t))(b(t) - x(L,t)), \tag{2.36}$$

with

$$b(t) = x(L, t) + (l^2 - y^2(L, t))^{\frac{1}{2}}. \tag{2.37}$$

The initial conditions (2.17) become

$$x(s, 0) = x_1(s), \tag{2.38}$$

$$y(s, 0) = y_1(s), \tag{2.39}$$

$$u(s, 0) = v(s, 0) = 0, \quad L_0 \leq s \leq L. \tag{2.40}$$

The dimensionless dynamic force E and recoil force P are given by

$$E(t) = -2\, m_a \ddot{b}, \tag{2.41}$$

and

$$P(t) = 2\,\{M'(L_0, t)\, y'_0(L_0) + T(L_0, t)\, x'_0(L_0)\}. \tag{2.42}$$

The finite-difference method discussed in the next section can be used for the solution of both the static and the dynamic equations. In [6] the static problem, which is a two-point boundary-value problem for a system of ordinary differential equations, is solved by means of a shooting method.

3. Finite difference equations

In order to obtain approximations for the solution of the partial differential equations (2.27-32), with boundary conditions (2.33-36) and initial conditions (2.38-40) we use a finite-difference method. We consider the grid

$$s = j\,\Delta s, \qquad j = 0(1)n_s,\ n_s\,\Delta s = L - L_0, \tag{3.1}$$

and

$$t = k\,\Delta t, \qquad k = 0(1)n_t, \tag{3.2}$$

n_t being an integer large enough to cover the time interval of interest. The grid points are indicated by 'x' in Figure 3.1. To satisfy the boundary conditions external mesh points are introduced, $(L_0\text{-}\Delta s,\ k\,\Delta t)$ and $(L + \Delta s,\ k\,\Delta t)$, with $k = 0(1)n_t$, indicated by 'Δ' and '$\square$', respectively. The value of a function $f(s, t)$ at the grid point $(j\,\Delta s,\ k\,\Delta t)$ is denoted by $f_{j,k}$ and of $h(s)$ and $g(t)$ by h_j and g_k, respectively.

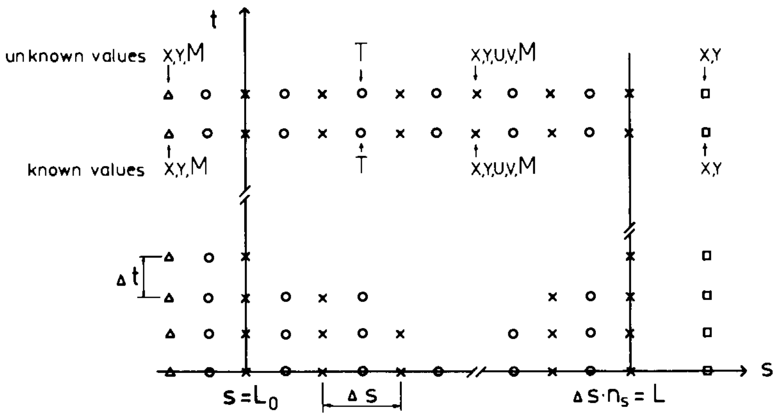

Figure 3.1. Grid placed over the s, t-plane.

There are many difference schemes possible to approximate the differential equations. For instance the term $(Tx')'(j\,\Delta s, k\,\Delta t)$ can be approximated by

$$T_{j,k}\,\frac{(x_{j+1,k}-2x_{j,k}-x_{j-1,k})}{\Delta s^2}+\frac{(T_{j+1,k}-T_{j-1,k})}{2\,\Delta s}\,\frac{(x_{j+1,k}-x_{j-1,k})}{2\,\Delta s}\,, \tag{3.3}$$

but also by

$$\left\{T_{j+\frac{1}{2},k}\,\frac{(x_{j+1,k}-x_{j,k})}{\Delta s}-T_{j-\frac{1}{2},k}\,\frac{(x_{j,k}-x_{j-1,k})}{\Delta s}\right\}/\,\Delta s. \tag{3.4}$$

In the last case the normal force T is defined at each time level only at points just in between the grid points (3.1), indicated by '0' in Figure 3.1. The normal force $T(L, t)$ at the tip of the bow, occurring in the boundary conditions (2.35) and (2.36), can at time $t = k\,\Delta t$ for instance be approximated by

$$\frac{3}{2}\,T_{n_s-\frac{1}{2},k}-\tfrac{1}{2}\,T_{n_s-1\frac{1}{2},k}\,. \tag{3.5}$$

The same kind of approximation (3.3) and (3.4) can be used for the other terms on the right-hand sides of (2.27) and (2.28). The constraint (2.31) can be approximated at the grid points, yielding for point $(j\,\Delta s, k\,\Delta t)$

$$\{(x_{j+1,k}-x_{j-1,k})/\,2\,\Delta s\}^2+\{(y_{j+1,k}-y_{j-1,k})/\,2\,\Delta s\}^2=1\,. \tag{3.6}$$

When we approximate this constraint at points in the middle of the grid points we obtain

$$\{(x_{j,k}-x_{j-1,k})/\,\Delta s\}^2+\{(y_{j,k}-y_{j-1,k})/\,\Delta s\}^2=1\,. \tag{3.7}$$

The type of approximation (3.4) in combination with (3.7) turned out to be satisfactory because it is well matched to the boundary conditions.

If we use (3.4), and take a weighted average by means of the factor μ, of forward and backward approximations of each of the four equations (2.27) ... (2.30) we find

$$\begin{aligned}
V_j(u_{j,k+1} - u_{j,k})/\Delta t = \\
&\mu\,[\{ T_{j+\frac{1}{2},k+1}\,(x_{j+1,k+1} - x_{j,k+1}) - T_{j-\frac{1}{2},k+1}\,(x_{j,k+1} - x_{j-1,k+1})\}/\Delta s^2 + \\
&\quad \{(M_{j+1,k+1} - M_{j,k+1})\,(y_{j+1,k+1} - y_{j,k+1}) - (M_{j,k+1} - M_{j-1,k+1}) \\
&\qquad (y_{j,k+1} - y_{j-1,k+1})\}/\Delta s^3\,] + \\
&(1-\mu)\,[\{ T_{j+\frac{1}{2},k}\,(x_{j+1,k} - x_{j,k}) - T_{j-\frac{1}{2},k}\,(x_{j,k} - x_{j-1,k})\}/\Delta s^2 + \\
&\quad \{(M_{j+1,k} - M_{j,k})\,(y_{j+1,k} - y_{j,k}) - (M_{j,k} - M_{j-1,k}) \\
&\qquad (y_{j,k} - y_{j-1,k})\}/\Delta s^3\,], \\
&\qquad j = 0(1)\,n_s - 1,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
V_j(v_{j,k+1} - v_{j,k})/\Delta t = \\
&\mu\,[\{ T_{j+\frac{1}{2},k+1}\,(y_{j+1,k+1} - y_{j,k+1}) - T_{j-\frac{1}{2},k+1}\,(y_{j,k+1} - y_{j-1,k+1})\}/\Delta s^2 - \\
&\quad \{(M_{j+1,k+1} - M_{j,k+1})\,(x_{j+1,k+1} - x_{j,k+1}) - (M_{j,k+1} - M_{j-1,k+1}) \\
&\qquad (x_{j,k+1} - x_{j-1,k+1})\}/\Delta s^3\,] + \\
&(1-\mu)\,[\{ T_{j+\frac{1}{2},k}\,(y_{j+1,k} - y_{j,k}) - T_{j-\frac{1}{2},k}\,(y_{j,k} - y_{j-1,k})\}/\Delta s^2 - \\
&\quad \{(M_{j+1,k} - M_{j,k})\,(x_{j+1,k} - x_{j,k}) - (M_{j,k} - M_{j-1,k}) \\
&\qquad (x_{j,k} - x_{j-1,k})\}/\Delta s^3\,], \\
&\qquad j = 0(1)\,n_s - 1,
\end{aligned} \tag{3.9}$$

$$(x_{j,k+1} - x_{j,k})/\Delta t = \mu u_{j,k+1} + (1-\mu)\,u_{j,k}, \quad j = 0(1)\,n_s, \tag{3.10}$$

$$(y_{j,k+1} - y_{j,k})/\Delta t = \mu v_{j,k+1} + (1-\mu)\,v_{j,k}, \quad j = 0(1)\,n_s. \tag{3.11}$$

Using (3.7) we approximate (2.31) and (2.32) by

$$\{(x_{j,k+1} - x_{j-1,k+1})/\Delta s\}^2 + \{(y_{j,k+1} - y_{j-1,k+1})/\Delta s\}^2 = 1, \qquad j = 0(1)\,n_s + 1 \tag{3.12}$$

and

$$M_{j,k+1} = W_j[\{(x_{j+1,k+1} - x_{j-1,k+1})/2\Delta s\}\{(y_{j+1,k+1} - 2y_{j,k+1} + y_{j-1,k+1})/\Delta s^2\} -$$
$$\{(y_{j+1,k+1} - y_{j-1,k+1})/2\Delta s\}\{(x_{j+1,k+1} - 2x_{j,k+1} + x_{j-1,k+1})/\Delta s^2\} +$$
$$+ \theta_0'(j\Delta s)], \quad j = 0(1)n_s. \tag{3.13}$$

For $\mu = \frac{1}{2}$ these equations become the Crank-Nicolson scheme and the truncation error is $O(\Delta t^2) + O(\Delta s^2)$. For $\mu = 1$ we have the fully implicit backward time difference scheme, then the truncation error is $O(\Delta t) + O(\Delta s^2)$.

The boundary conditions (2.33) are approximated by

$$x_{0,k+1} = x_0(L_0), \quad y_{0,k+1} = y_0(L_0), \tag{3.14}$$

and

$$(y_{1,k+1} - y_{-1,k+1})/2\Delta s = y_0'(L_0). \tag{3.15}$$

Before writing down the boundary conditions at $s = L$ we mention that, besides the x-coordinate b of the arrow, it appeared to be advantageous to introduce also its velocity

$$c \overset{\text{def}}{=} \dot{b} \tag{3.16}$$

as another unknown function. Then the three boundary conditions (2.34-36) at $s = L$ are approximated by the difference relations

$$M_{n_s,k+1} = 0, \tag{3.17}$$

$$m_a(c_{k+1} - c_k)/\Delta t + m_t(u_{n_s,k+1} - u_{n_s,k})/\Delta t =$$
$$- \mu[\{\frac{3}{2} T_{n_s-\frac{1}{2},k+1} - \tfrac{1}{2} T_{n_s-1\frac{1}{2},k+1}\}\{(x_{n_s+1,k+1} - x_{n_s-1,k+1})/2\Delta s\} +$$
$$\{(y_{n_s+1,k+1} - y_{n_s-1,k+1})/2\Delta s\}\{(-4M_{n_s-1,k+1} + M_{n_s-2,k+1})/2\Delta s\}]$$
$$- (1-\mu)[\{\frac{3}{2} T_{n_s-\frac{1}{2},k} - \tfrac{1}{2} T_{n_s-1\frac{1}{2},k}\}\{(x_{n_s+1,k} - x_{n_s-1,k})/2\Delta s\} +$$
$$\{(y_{n_s+1,k} - y_{n_s-1,k})/2\Delta s\}\{(-4M_{n_s-1,k} + M_{n_s-2,k})/2\Delta s\}], \tag{3.18}$$

$$(\mu y_{n_s,k+1} + (1-\mu)y_{n_s,k})m_a(c_{k+1} - c_k)/\Delta t - \{\mu(b_{k+1} - x_{n_s,k+1}) +$$
$$+ (1-\mu)(b_k - x_{n_s,k})\}m_t(v_{n_s,k+1} - v_{n_s,k})/\Delta t =$$
$$[\mu(b_{k+1} - x_{n_s,k+1}) + (1-\mu)(b_k - x_{n_s,k})]$$
$$[\mu[\{\frac{3}{2} T_{n_s-\frac{1}{2},k+1} - \tfrac{1}{2} T_{n_s-1\frac{1}{2},k+1}\}\{(y_{n_s+1,k+1} - y_{n_s-1,k+1})/2\Delta s\} -$$

$$\{(x_{n_s+1,k+1} - x_{n_s-1,k+1})/2\Delta s\}\{(-4M_{n_s-1,k+1} + M_{n_s-2,k+1})/2\Delta s\}]$$

$$+ (1-\mu)[\{\frac{3}{2} T_{n_s-\frac{1}{2},k} - \tfrac{1}{2} T_{n_s-1\frac{1}{2},k}\}\{(y_{n_s+1,k} - y_{n_s-1,k})/2\Delta s\} -$$

$$\{(x_{n_s+1,k} - x_{n_s-1,k})/2\Delta s\}\{(-4M_{n_s-1,k} + M_{n_s-2,k})/2\Delta s\}]]. \tag{3.19}$$

Finally we take as approximations for (2.37) and (3.16)

$$(b_{k+1} - x_{n_s,k+1})^2 + y^2_{n_s,k+1} = l^2, \tag{3.20}$$

and

$$(b_{k+1} - b_k)/\Delta t = \mu c_{k+1} + (1-\mu) c_k, \tag{3.21}$$

respectively. The dynamic force E (2.41) is approximated by

$$E_{k+1} = -2m_a (c_{k+1} - c_k)/\Delta t, \tag{3.22}$$

and the recoil force P (2.42) by

$$P_{k+1} = 2\{(M_{1,k+1} - M_{-1,k+1})/2\Delta s \cdot y_0'(L_0) + \tfrac{1}{2}(T_{\frac{1}{2},k+1} + T_{-\frac{1}{2},k+1}) x_0'(L_0)\}. \tag{3.23}$$

At $t = 0$ the initial values of the unknown functions x, y, u, v are given by (2.38-40). The finite-difference approximation for the static equations can be found in a straightforward way from equations (3.8-21).

At each time, hence for each $k\Delta t$ ($k = 0(1)n_t$), we have to solve a set of nonlinear equations, which is done by means of a Newtonian method. For this method it is essential to have reliable starting values for the unknowns.

i) *The equations for the static case, for $t = 0$.*
Starting values for the solution of the static finite-difference equations are obtained by using the values computed by means of the program described in [6]. The reason that we revise these values by means of the static finite-difference scheme, is that the values obtained in this way are better matched to the finite-difference scheme for the dynamic equations.

ii) *The dynamic case, from $t = 0$ to $t = \Delta t$ ($k = 0$).*
We use as starting values of the unknowns at time level Δt the values obtained in i). In order to avoid the use of the values of the accelerations at $t = 0$ we take $\mu = 1$. In the Appendix we return to this.

iii) *The dynamic case, from $t = k\Delta t$ to $t = (k+1)\Delta t$ ($k = 1(1)n_t$).*
The starting values for the time level $(k+1)\Delta t$ of the unknowns x, y, u, v, M, b and c are obtained from the equations (3.8-11), (3.13), (3.17-19) and (3.21), with $\mu = 0$. This means that we explicitely calculate these values from the final results at the preceding time level $k\Delta t$. From these starting values we calculate the values at the time level $(k+1)\Delta t$, with $\mu = \frac{1}{2}$. Hence the further dynamical development for $t > \Delta t$ is determined by a Crank-Nicolson scheme.

In order to get accurate information about the way in which the arrow leaves the string, the mesh width Δt in the t-direction is chosen continuously smaller from a certain time, at which the string is nearly stretched. Because the difference scheme is a two-time-level one with approximations for only first-order derivatives with respect to time, no special provisions are needed.

For instance, in [7] numerical methods to solve related problems are analysed. In the nonlinear case only for specific problems stability and convergence of some difference schemes can be proved analytically. Here no proof is given of the stability and convergence of our difference scheme, however, we have checked our method numerically. First, in the static case, we compare automatically (see (i)) the results of the finite-difference method with the results obtained with the program described in [6]. The difference between the weight of the bow computed by both programs appears to be about .5%, if we take $n_s = 64$. Second, the total energy $\bar{A}_p + \bar{A}_k$ (equations (2.4) and (2.5)) has to be constant during the motion. Third, we can investigate the convergence of the difference equations by refining the grid. We consider the special bow

$$B(91.44,\ 10.16,\ \bar{W}(\bar{s}),\ \bar{V}(\bar{s}),\ \theta_0 \equiv 0,\ .0125{,}0,\ |\overline{OH}| = 15.24;\ 71.12,\ 15.53,\ .1625). \tag{3.24}$$

The bending stiffness $\bar{W}(\bar{s})$ and the mass distribution $\bar{V}(\bar{s})$ are given by

$$\bar{W}(\bar{s}) = 1.30 \cdot 10^5 \left(\frac{\bar{L} - \bar{s}}{\bar{L}}\right), \tag{3.25}$$

and

$$\bar{W}(\bar{s}) = 7.69 \text{ if } 1.30 \cdot 10^5 \left(\frac{\bar{L} - \bar{s}}{\bar{L}}\right) \leq 7.69, \tag{3.26}$$

$$\bar{V}(\bar{s}) = 4.52 \cdot 10^{-3} \left(\frac{\bar{L} - \bar{s}}{\bar{L}}\right). \tag{3.27}$$

The value at the tip of $\bar{W}(\bar{s})$ (3.26) is necessary in order to avoid difficulties in the calculation. This bow (H bow) is also discussed by Hickman in [1], page 69.

TABLE 3.1.

Dependence of $\bar{b}$, $\bar{c}$, $\bar{a}$ and $\bar{A}_p + \bar{A}_k$ on $\Delta\bar{t}$, $\Delta\bar{s}$ = 1.27 cm, $\bar{t}$ = .0157 sec.

$\Delta\bar{t}$ sec	$\bar{b}$ cm	$\bar{c}$ cm/sec	$\bar{a}$ cm/sec²	$\bar{A}_p + \bar{A}_k$ kgfcm
$4.9089 \cdot 10^{-4}$	16.379	−5544	−147704	560.48
$2.4544 \cdot 10^{-4}$	16.375	−5548	−139432	560.43
$1.2272 \cdot 10^{-4}$	16.373	−5549	−132578	560.41

TABLE 3.2.

Dependence of $\bar{b}$, $\bar{c}$, $\bar{a}$ and $\bar{A}_p + \bar{A}_k$ on $\Delta\bar{s}$, $\Delta\bar{t}$ = $1.2272 \cdot 10^{-4}$ sec, $\bar{t}$ = .0157 sec.

$\Delta\bar{s}$ cm	$\bar{b}$ cm	$\bar{c}$ cm/sec	$\bar{a}$ cm/sec²	$\bar{A}_p + \bar{A}_k$ kgfcm
5.08	16.06	−5583	−136392	569.8
2.54	16.24	−5563	−132585	563.6
1.27	16.37	−5549	−132578	560.4

In Tables 3.1 and 3.2 we show the dependence of some calculated dynamic quantities on the mesh widths Δt and Δs, respectively. The quantities are the x-coordinate $\bar{b}$ (cm) of the end of the arrow, the velocity $\bar{c} = \dot{\bar{b}}$ (cm/sec), the acceleration $\bar{a} = \dot{\bar{c}}$ (cm/sec^2) and the energy $\bar{A}_p + \bar{A}_k$ (kgfcm). The values are given for a fixed time $\bar{t} = .0157$ sec, which is near to the time at which the arrow leaves the string (.01662 sec). The same can be done for other times, then the results are similar with respect to convergence. From these tables it seems reasonable that with decreasing values of Δt and Δs the solutions of the difference equations 'converge'. The energy for $\Delta \bar{t} = 1.2272 \cdot 10^{-4}$ sec and $\Delta \bar{s} = 1.27$ cm differs about .5% from its value 557.207 kgfcm at time $\bar{t} = 0$.

A fourth check is to compare our results with those obtained by the use of the finite-element program MARC of the MARC Analysis Research Corporation. This has been done for the bow

$$B(91.44, 10.16, \bar{W}, \bar{V}, \theta_0 = 0, .01134, 0, \bar{l} = 89.34; 70.98, 15.43, .1589), \tag{3.28}$$

where the bending stiffness $\bar{W}(\bar{s})$ and the mass distribution $\bar{V}(\bar{s})$ are given by

$$\bar{W}(\bar{s}) = 1.15 \cdot 10^5 \left(\frac{\bar{L} - \bar{s}}{\bar{L} - \bar{L}_0}\right), \tag{3.29}$$

and

$$\bar{W}(\bar{s}) = 7.69 \text{ if } 1.15 \cdot 10^5 \left(\frac{\bar{L} - \bar{s}}{\bar{L} - \bar{L}_0}\right) \leq 7.69, \tag{3.30}$$

$$\bar{V}(\bar{s}) = 3.91 \cdot 10^{-3} \left(\frac{\bar{L} - \bar{s}}{\bar{L} - \bar{L}_0}\right). \tag{3.31}$$

TABLE 3.3.

Comparison between finite-difference and finite-element solution.

	finite element		finite difference	
$\bar{t}$ sec	$\bar{b}$ cm	$\bar{c}$ cm/sec	$\bar{b}$ cm	$\bar{c}$ cm/sec
$.501 \cdot 10^{-2}$	63.69	−2739	63.53	−2795
$1.001 \cdot 10^{-2}$	45.47	−4399	44.98	−4480
$1.401 \cdot 10^{-2}$	25.69	−5449	24.84	−5537

In the MARC program the functions $\bar{W}$ and $\bar{V}$ are approximated by step functions and both the bow and the string are taken slightly extensible. The number of elements used was eight, and $\Delta \bar{t} = .001$ sec. For the finite-difference scheme we used $\Delta \bar{t} = .001$ sec and $\Delta \bar{s} = 1.27$ cm. The values of $\bar{b}$ and $\bar{c}$ are given in Table 3.3 for several values of $\bar{t}$. In Figure 3.2 the acceleration $\bar{a}$ of the arrow in cm/sec^2 as function of the time in sec, computed by both programs is drawn. We conclude that there is a reasonable agreement between the results with respect to the x-coordinate $\bar{b}$ and the velocity $\bar{c}$ of the arrow. The acceleration curve obtained by using the MARC-program, however, is oscillating in a non-physical way.

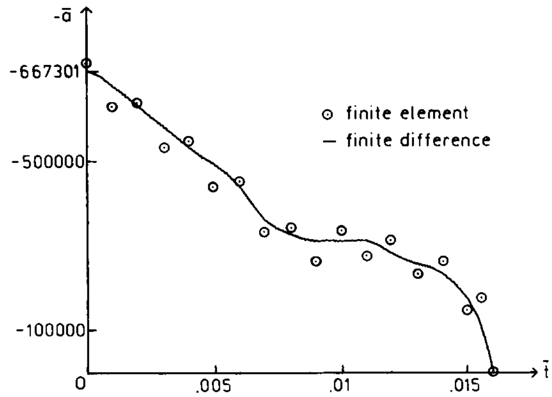

Figure 3.2. Accelaration of arrow.

4. Some numerical results

In [6] the so called static quality coefficient, denoted by q, was defined. This quantity is given by

$$q = \bar{A}/\{|\overline{OD}| \cdot \bar{F}(|\overline{OD}|)\}, \tag{4.1}$$

where $\bar{A}$ is the energy stored in the bow by deforming it from the braced position into the fully drawn position. This energy reads

$$\bar{A} = \int_{|\overline{OH}|}^{|\overline{OD}|} \bar{F}(\bar{b})\,\mathrm{d}\bar{b} = \left[\int_{\bar{L}_0}^{\bar{L}} \bar{W}(\bar{s})\ (\theta'(\bar{s}) - \theta'_0(\bar{s}))^2\,\mathrm{d}\bar{s} \right]_{\bar{b}=|\overline{OH}|}^{\bar{b}=|\overline{OD}|} \tag{4.2}$$

We now introduce two dynamic quality coefficients η and ν in order to be able to compare more easily the performance of different bows in combination with various arrows. The efficiency η of a bow is defined by

$$\eta = \bar{m}_a\,\bar{c}_\ell^2/\bar{A}, \tag{4.3}$$

where $\bar{c}_\ell$ is the muzzle velocity. The produkt $q.\eta$ is a measure for the energy imparted to the arrow. It is evident that in all kinds of archery we want this quantity to be as large as possible. However, it cannot be on its own an appropriate measure of the performance of the bow. If we let for instance increase the mass m_a of the arrow indefinitely, then the efficiency (4.3) tends to its largest value, namely one, hence $q\eta$ tends to its largest value q, however, the muzzle velocity c_ℓ tends to zero. Klopsteg [1], page 162, mentioned the cast as another criterion of the quality of a bow. He defines the cast as the property of a bow that enables it to impart velocity to an arrow of stated mass. So a second dynamic quality coefficient can be defined by

$$\nu = \left(\frac{q\eta}{m_a}\right)^{\frac{1}{2}} = c_\ell, \tag{4.4}$$

where the last equality follows from (4.1), (4.3) and (2.26). Thus, if the weight, draw and mass of the limbs are stated, then ν is a measure for the muzzle velocity of a given arrow. In order to show on which dimensionless quantities ν depends, we write

$$\nu = c_{\ell}(L, L_0, W(s), V(s), \theta_0(s), m_a, m_t, |OH| \text{ or } \ell), \quad 0 \leq s \leq L. \tag{4.5}$$

For flight shooting the quality coefficient ν is important because then ν is wanted to be sufficiently large. For hunting (but certainly for target shooting) it is not easy to find a criterion for the good performance of bow and arrow. What we can state is that the bow has to shoot 'sweetly' and without an unpleasant recoil. By this we mean that the acceleration of the arrow should happen smoothly enough and that the recoil force P (2.42) should be not too large or fluctuating too strongly.

One of our objectives is to get insight into the influence of the quantities which determine a bow on the numbers q, η and ν, and on the behaviour of the forces $E(b)$ and $P(b)$. To this end we start with the H bow

$$B(1.286, .143, W(s), V(s), \theta_0 \equiv 0, .0769, 0, |OH| = .214; 1,1,1), \tag{4.6}$$

and change in a more or less systematic way its parameters. The bending stiffness W and the mass distribution V in (4.6) are given by (3.25), (3.26) and (3.27) of which the values have been made dimensionless by using (2.26).

If the three quantities q, η and ν are known the muzzle velocity $\bar{c}_{\ell}$ (cm/sec) can be computed. Using (4.4) and (2.26) we find

$$\bar{c}_{\ell} = 31.32\, \nu \left(\frac{\bar{F}(|\overline{OD}|) \cdot |\overline{OD}|}{\bar{m}_b} \right)^{\frac{1}{2}} \text{cm/sec}, \tag{4.7}$$

where the number 31.32 is caused by the choice of our units. The kinetic energy (kgfcm) imparted to the arrow of mass 2 $\bar{m}_a$ follows from

$$\bar{m}_a \bar{c}_{\ell}^2 = m_a \nu^2 \bar{F}(|\overline{OD}|) \cdot |\overline{OD}| = \eta q \bar{F}(|\overline{OD}|) \cdot |\overline{OD}|. \tag{4.8}$$

These equations show the dependence of the two important quantities, the muzzle velocity (4.7) and the kinetic energy of the arrow (4.8), on the weight, draw and mass of the limbs. For the H bow (3.24) we have $\bar{F}(|\overline{OD}|) = 15.53$ kgf, $|\overline{OD}| = 71.1$ cm and $\bar{m}_b = .1625$ kg, and the computed values of q, η and ν are .407, .89 and 2.16, respectively. Thus for this bow, $\bar{c}_{\ell} = 5578$ cm/sec and $\bar{m}_a \bar{c}_{\ell}^2 = 400$ kgfcm. The shooting time (the time interval between the loosing of the arrow and its leaving the string) appeared to be .01662 sec.

In Fig. 4.1 we have drawn the dimensionless static-force-draw (SFD) curve $F(b)$, and dynamic-force-draw (DFD) curve $E(b)$, calculated by our theory for the H bow. Also the curves obtained by Hickman's theory [1], page 69, are drawn. The SFD-curves coincide with each other within drawing accuracy. As can be seen from Figure 4.1 the DFD curve obtained by using the finite-difference method is gradually decreasing. There is no jump in the force on the arrow at $t = 0$. The finite-difference method will in general give an efficiency which is smaller than one.

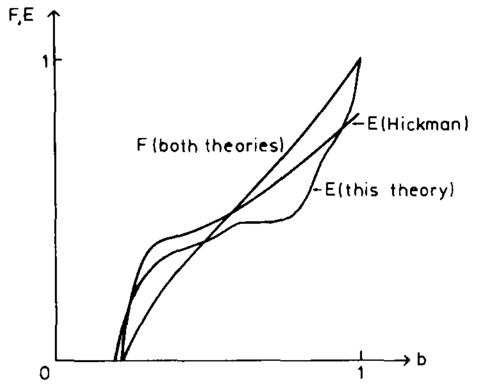

Figure 4.1. SFD and DFD curves, Hickman's theory and this theory.

The x-coordinate b of the arrow for which the force at the arrow is zero, hence the value of b where the arrow leaves the string, is a bit smaller than the brace height.

Hickman used a simple model (H model) which consists of two rigid limbs without mass, connected each to the grip by means of a linear elastic hinge. The strength of these hinges is determined in some way by the elastic properties of the real bow. The mass of the real limbs is accounted for by concentrated masses at the tips of the limbs. Because of these masses the force on the arrow has, when calculated by means of the H model, a jump at time $t = 0$. In [6] it is proved that the efficiency η of a H model bow is always 1. That is why in Figure 4.1 the area below the SFD curve and the area below the DFD curve, calculated by means of the H model, are equal.

We mention that in the figures given by Hickman in [1], page 5 and 7, the acceleration of the arrow measured experimentally, and hence also the force on the arrow, is zero at time $t = 0$, which is in contradiction with his own model. The dynamic force on the arrow in our theory at that moment is, if there are no concentrated masses at the tips ($m_t = 0$), equal to the static force in fully drawn position (see Appendix).

The shapes of the limbs of the H bow for some positions of the arrow, both static and dynamic, are shown in Figure 4.2. For $b = 1$ both shapes are the same. After loosing the arrow first the outer parts of the limbs stretch themselves. The released bending energy is used to accelerate both the arrow and the limbs. For a certain value of b the shape in the dynamic and static case are nearly the same. After that the outer parts of the limbs are decelerated and become more sharply bent than in the static case. Now the inner parts of the limbs become more stretched and loose their bending energy.

In Figure 4.3 the DFD curve and the recoil force P, as a function of the position of the end of the arrow b, are drawn. It can be seen that although the force E at the arrow decreases after release of the arrow, the recoil force P increases and becomes more than two times the weight of the bow. We note that at a certain moment it becomes negative; this means that the archer has to pull instead of to push the bow at the end of the shooting in order to keep the grip at its place. In modern archery, however, it is practice to shoot open-handed. But then it is impossible for an archer to exert a force on the bow directed to himself and the assumption that the bow is clamped at the grip, is violated. Possibly less kinetic energy will be recovered from the bow

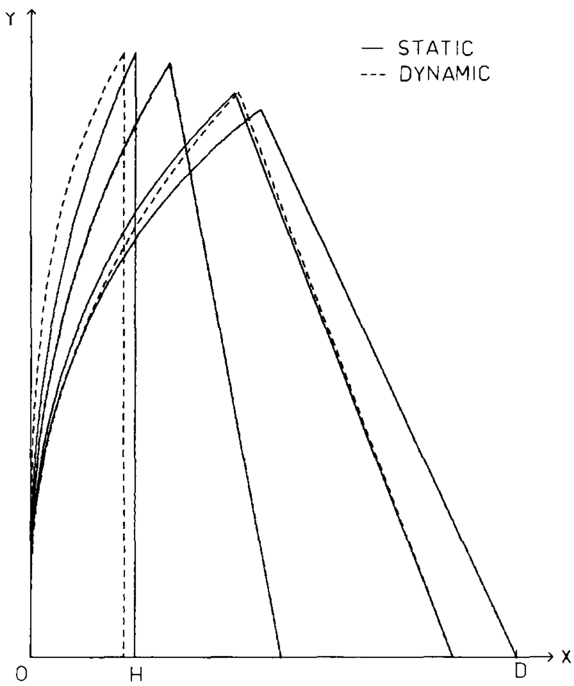

Figure 4.2. Shapes of limbs of H bow.

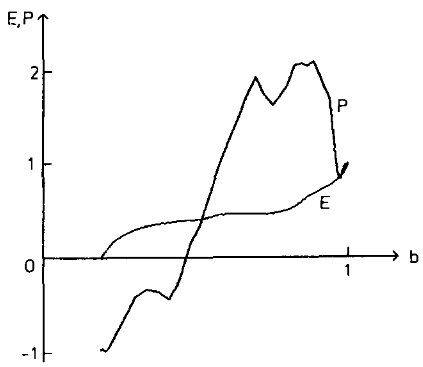

Figure 4.3. Dynamic force $E(b)$ and recoil force $P(b)$ for the H bow.

when negative recoil forces occur if the bow is shot open-handed. In this paper we adhere to the assumption that the grip of the bow is clamped.

Klopsteg [1], page 141, carried out experiments to investigate the motion of the bow hand while the arrow is being accelerated. He finds, besides other movements, always a small excursion of this hand backwards after the loose. He states: 'A satisfactory explanation for the slight backward motion is that during the 20 or 30 thousands of a second after the loose, a very considerable force is being exerted by the string on the arrow and consequently an equal backward force is exerted by the handle of the bow on the bow hand. During this brief impulse the instantaneous value of the force may rise to several hundred pounds, but lasting for an exceedingly few thousands of a second'. This explanation is in contradiction with the results shown in Figure 4.3. The dynamic force E at the arrow and the force P at the bow hand are not equal at all.

In what follows we consider the consequences of a change of one characteristic quantity of the H bow at a time, the other ones being kept the same. The values of the static quality coefficient q given in the following tables are computed by means of the program described in [6]. Only if the smoothness of the DFD curve or the behaviour of the recoil force P differs clearly from the smoothness of that curve in the case of the H bow, this is explicitly mentioned.

The influence of a change of the length of the grip $2L_0$ is shown in Table 4.1. In [1], page 18, the effect of a rigid middle section, a grip, is also dealt with. This is an interesting subject because it is known that a bow which bends throughout its whole length is not a pleasant bow to shoot. It has a so-called 'kick'. Because Hickman did not found striking theoretical differences with respect to the static properties of two bows, one with $L_0 = 0$ and the other with $L_0 = .143$, he states: 'The greatest difference between these two types of bows is due to dynamical conditions'. However, it is seen from Table 4.1 that the values of q, η and ν nearly do not change. From our calculations it follows that the behaviour of the dynamic force E and of the recoil

TABLE 4.1.

Influence of grip length $2L_0$.

L_0	0	.0714	.143	.214
q	.415	.411	.407	.403
η	.88	.88	.89	.90
ν	2.18	2.18	2.16	2.16

force P, are almost the same for the two types. Hence also with respect to these dynamic properties no clear differences appear in our theory.

In Table 4.2 the influence of the brace height is shown. In [1], page 21, Hickman makes the following statement based on experiments: 'The arrow velocity increases with increase in bracing height up to a certain point, after which it slowly decreases with additional increases in bracing height. The bracing height for maximum arrow velocity depends principally on the length of the bow'. This does not agree with the results of our theory. From Table 4.2 we see that there is always a small decrease of the arrow velocity when the brace height is increased. This is due to both static (q) and dynamic (η) effects.

To investigate the influence of the length $2L$ of the bow we considered five different lengths. From Table 4.3 we find that there is almost no perceptible change in the efficiency η of the bow, hence ν shows the same tendency as q.

TABLE 4.2.

Influence of brace height |OH|.

|OH|	.0714	.107	.143	.179	.214	.250
q	.444	.438	.430	.420	.407	.392
η	.91	.91	.90	.89	.89	.88
ν	2.29	2.28	2.25	2.21	2.16	2.12

TABLE 4.3.

Influence of length 2L.

L	1.143	1.214	1.286	1.357	1.429
q η ν	.393 .88 2.12	.400 .88 2.15	.407 .89 2.16	.413 .89 2.18	.417 .89 2.21

Now we consider the influence of the distribution of the bending stiffness W and the mass V along the bow. We take

$$W_n(s) = 1.47 \left(\frac{L - s}{L - L_0}\right)^{\beta_n}, \quad L_0 \leq s \leq L, \tag{4.9}$$

and

$$V_n(s) = 1.75 \left(\frac{L - s}{L - L_0}\right)^{\beta_n}, \quad L_0 \leq s \leq L, \tag{4.10}$$

where $n = 1, 2, 3, 4$ and $\beta_1 = 0, \beta_2 = \frac{1}{2}, \beta_3 = 1, \beta_4 = 2$. A value of β_n chosen in (4.10) needs not to be the same as the one chosen in (4.9). In order to avoid numerical difficulties we take again

$$W_n(s) = 10^{-4} \text{ if } 1.47 \left(\frac{L - s}{L - L_0}\right)^{\beta_n} \leq 10^{-4}. \tag{4.11}$$

The results of changing W and V separately are given in Table 4.4. We conclude that if the mass distribution V is taken to be linear (V_3), the constant bending stiffness distribution (W_1) is the best, due to both static (q) and dynamic (η) effects. If the bending stiffness W is linear (W_3) then the mass distribution V_4, which has light tips, is undoubtedly the best. We refer to Figure 4.4 for its DFD curve.

TABLE 4.4.

Influence of bending stiffness W and of mass V.

W(s)	W_1	W_2	W_3	W_3			
V(s)		V_3		V_1	V_2	V_3	V_4
q η ν	.417 .93 2.25	.414 .91 2.21	.407 .89 2.16	.407 .74 1.98	.407 .81 2.08	.407 .89 2.16	.407 .97 2.28

TABLE 4.5.

Influence of bending stiffness W and of mass V.

W,V	W_1, V_1	W_2, V_2	W_3, V_3
q η ν	4.17 .87 2.16	.414 .86 2.15	.407 .89 2.16

In Table 4.5 W and V are changed simultaneously. The results in this table show that the quantities q, η and ν of the bow nearly depend only on the ratio of the two functions W and V. However, as can be seen from Figure 4.4, the DFD curve of the bow with W_3 and V_3 is far more smooth than those of the other two bows. It shows that although the efficiency of a bow with uniform distributions of bending stiffness and mass is acceptable, it will shoot almost surely unpleasant.

We now consider the influence of the shape of the bow in unbraced situation. This shape is determined by the function $\theta_0(s)$. We choose

$$\theta_{0,1}(s) = 0, \; 0 \leq s \leq L_0, \; \theta_{0,1}(s) = -.12 \qquad , L_0 \leq s \leq L, \tag{4.12}$$

$$\theta_{0,2}(s) = 0, \; 0 \leq s \leq L_0, \; \theta_{0,2}(s) = -.5\,(s-L_0), \quad L_0 \leq s \leq L, \tag{4.13}$$

$$\theta_{0,3}(s) = 0, \; 0 \leq s \leq L_0, \; \theta_{0,3}(s) = .12 - (s-L_0), \quad L_0 \leq s \leq L. \tag{4.14}$$

The three forms are drawn in Figure 4.5. The H bow in unbraced situation is straight, hence it is part of the y-axis, $\theta_0 \equiv 0$. The unbraced situations (4.12), (4.13) and (4.14) are called to possess recurve as we mentioned before. We have to choose a moderate recurve in order to agree

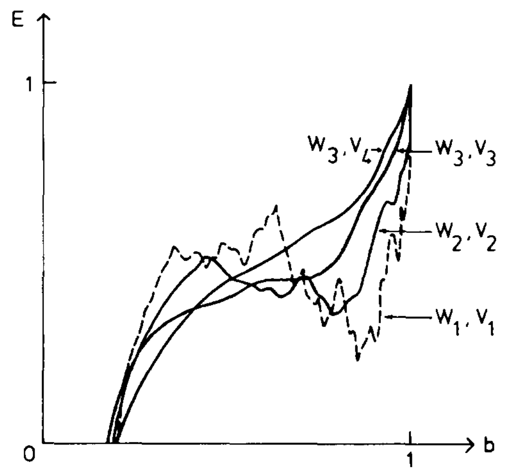

Figure 4.4. DFD curves for bows (W_1, V_1), (W_2, V_2), (W_3, V_3) and (W_3, V_4).

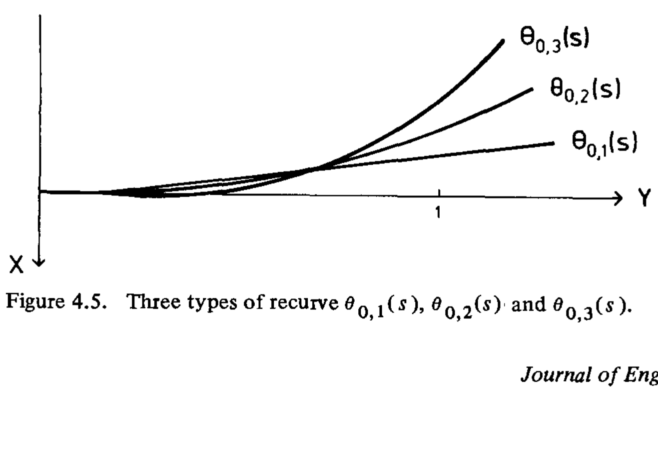

Figure 4.5. Three types of recurve $\theta_{0,1}(s)$, $\theta_{0,2}(s)$ and $\theta_{0,3}(s)$.

with the assumption that the string has contact with the bow only at the tips of the bow. It is seen from Table 4.6 that the efficiency of the recurved bows is slightly smaller than the efficiency of the H bow ($\theta_0 \equiv 0$). In the case of $\theta_{0,3}$ however, there is a more important favourable influence of the recurve on the static quality coefficient q. This agrees with the experience of Hickman [1], pages 22, 24 and 50. In [6] a bow with even a coefficient q equal to .833 is described. However, for this bow the string lies partly along the bow during some time interval. In a following paper we hope to be able to describe the dynamic performance of such a bow.

TABLE 4.6.
Influence of shape of unbraced bow.

θ_0	$\theta_0 \equiv 0$	$\theta_{0,1}$	$\theta_{0,2}$	$\theta_{0,3}$
q	.407	.424	.457	.487
η	.89	.83	.81	.83
ν	2.16	2.14	2.19	2.29

We stress that for a bow with a shape given by $\theta_{0,3}$ the recoil force P at the bow hand is positive at all times in between loosing the arrow and its leaving the string (Figure 4.6). This is in contradiction to all other bows mentioned so far.

Next the influence of the mass of the arrow is considered. In Table 4.7 the consequences of changing m_a are collected. Now also the product of q and η is given, being a measure of the energy imparted to the arrow. The factor q is .407 in all cases. The first and last given arrow masses in Table 4.7 are of little practical importance, however, they show what happens in the case of a light or heavy arrow. When the mass of the arrow is somewhat smaller than the smallest mass mentioned in this table the force exerted on the arrow by the string becomes zero before the string is stretched and hence our theory may be no longer valid. We remark that the de-

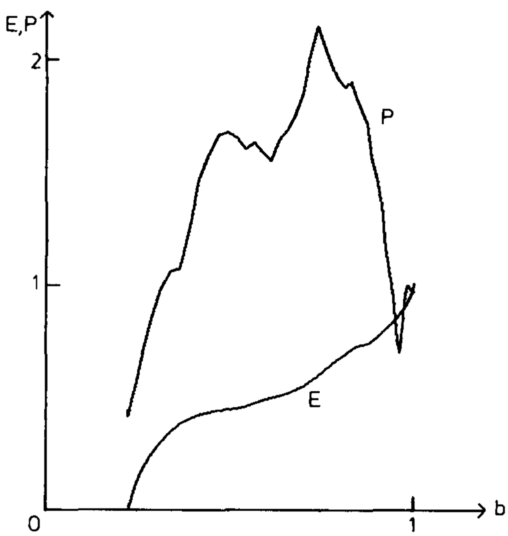

Figure 4.6. Dynamic force $E(b)$ and recoil force $P(b)$ for recurve $\theta_{0,3}$.

TABLE 4.7.
Influence of mass of arrow $2m_a$, $q = .407$.

m_a	.0192	.0384	.0769	.1538	.3077
η	.48	.69	.89	.98	.98
$q \cdot \eta$	.20	.28	.36	.40	.40
ν	3.20	2.72	2.16	1.63	1.14

crease of the efficiency with the decrease of the arrow mass, shown in Table 4.7, does not occur in the H model. Table 4.7 shows further that although the efficiency of a bow shooting a light arrow is bad, the muzzle velocity will be high, a fact already mentioned in many books about archery.

In [1], page 167, Klopsteg defines the concept of virtual mass as: 'A mass which, if it were moving with the speed of the arrow at the instant the latter leaves the string, would have precisely the kinetic energy of the limbs and the string at that instant'. If $\bar{K}_h$ denotes the half of the virtual mass then

$$\bar{A} = (\bar{m}_a + \bar{K}_h)\, \bar{c}_\ell^{\,2}\,. \tag{4.15}$$

If we define $K_h = \bar{K}_h / \bar{m}_b$, we obtain by using (4.3)

$$K_h = m_a \left(\frac{1 - \eta}{\eta} \right). \tag{4.16}$$

Klopsteg continues: 'That the virtual mass is in fact a constant, has been determined in many measurements with a large number of bows'. However, if we compute K_h using (4.16) for three values of m_a, .0384, .0769, .1538, we get .017, .010, .003, respectively. So by our theory K_h is definitely not independent of the mass of the arrow in the case of the H bow.

In Figure 4.7 we depict one SFD curve and a number of DFD curves for different values of

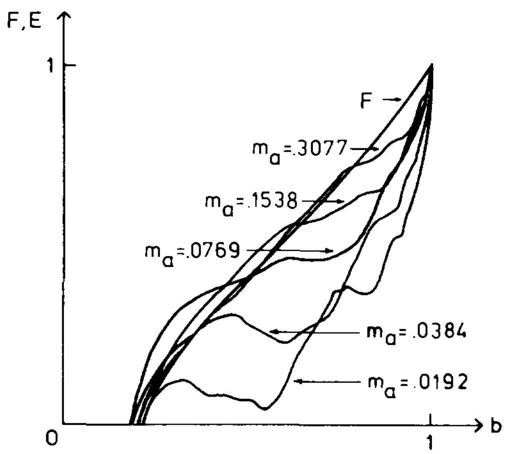

Figure 4.7. SFD curve $F(b)$ and DFD curves $E(b)$ of H bow, different arrow masses $2m_a$, Table 4.7.

m_a. If the mass m_a becomes larger the DFD curve approaches the SFD curve. With respect to the maximum value of the recoil force P we note that, if m_a tends to infinity, we get a quasi-static situation and hence also P as a function of b will follow closely the SFD curve. It appeared that the maximum value of P increases if the mass of the arrow decreases. For $m_a = .0192$ we even get a maximum value of P equal to about 5 times the weight of the bow.

Finally the influence of concentrated masses m_t at each of the tips of a bow is described. For that purpose we give the parameter m_t three different, non-zero values. In Table 4.8 the

TABLE 4.8.

Influence of concentrated tip masses m_t, $q = .407$.

m_t	0	.0769	.1538	.2307
η ν	.89 2.16	.87 2.15	.84 2.11	.82 2.08

value of q is .407 in all cases. From this table it follows that the efficiency decreases slightly if the mass m_t at the tips increases. In Figure 4.8 the DFD curves are drawn. It is seen that the force E on the arrow possesses a jump at the time $t = 0$. This jump becomes larger when m_t increases. Most of the energy used to accelerate at early instants the concentrated masses at the tips is transferred later on to the arrow. This follows from the fact that the forces on the arrow grow with increasing values of m_t in the region where the string becomes more stretched.

In [1], page 47, Hickman describes an experiment made to find the effect of the mass at the bow tips. We quote: 'Measurements of velocities for different weight arrows showed that a load of 400 grain (.02592 kg) added to the arrow weight, reduced the velocity by about 42 feet per second or 25 percent. In contrast to this, the same load added to the tips only reduced the velocity even for a light arrow, by about $1\frac{1}{2}$ feet per second or approximately one percent'. From Table 4.7, third and fourth column, it follows that if we increase the half arrow mass m_a by .0769, the velocity decreases by 24.8 percent. From Table 4.8, first and second column, it follows that if we add a mass $m_t = .0769$ to each of the tips the velocity decreases only by .7 per-

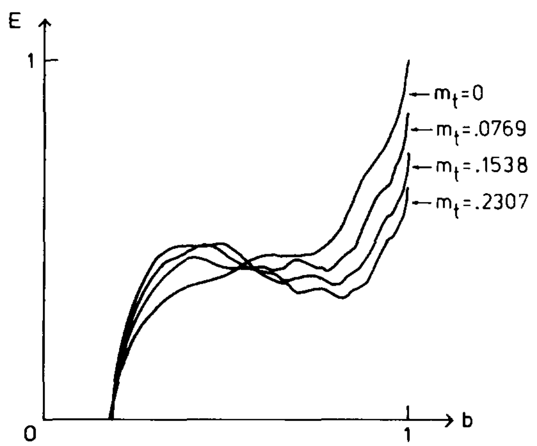

Figure 4.8. DFD curves for H bow with masses m_t at the tips, Table 4.8.

cent. Although we do not know what type of bow Hickman used for his experiment, his findings agree qualitatively with these results.

APPENDIX

On the behaviour of the normal force T at $t = 0$

In this appendix we discuss the behaviour of the normal or longitudinal force T in the bow at the time the arrow is released, $t = 0$. In an early attempt we took for the first time step, from $t = 0$ to $t = \Delta t$ in the finite-difference scheme (Section 3) $\mu = \frac{1}{2}$. For the initial values of the unknown x, y, M and T we took their values in the static fully drawn position. When masses $m_t \neq 0$ were present at the tips we found that the resulting solution has a non-physical oscillatory character, indicating that the initial values for the unknowns were not sufficiently accurate. To improve the procedure, a fully implicit backward-time difference scheme ($\mu = 1$) for the first step ($k = 0$) is chosen (Section 3, (ii)). In this way the initial values of the normal force T are not used. We will now show that the static values of T cannot be used with respect to our method as initial values, when concentrated masses at the tips are present.

In Figure A.1 the normal force $T(L, t)$ at the tip is drawn as a function of time, for a very small time interval after the release of the arrow. This normal force is calculated by the method described in Section 3, for the bow

$$B(91.49,\ 10.16,\ \bar{W}(\bar{s}),\ \bar{V}(\bar{s}),\ \theta_0 \equiv 0,\ .0125,\ .0125,\ |\overline{OH}| = 15.29;$$
$$71.12,\ 15.53,\ .1625), \tag{A.1}$$

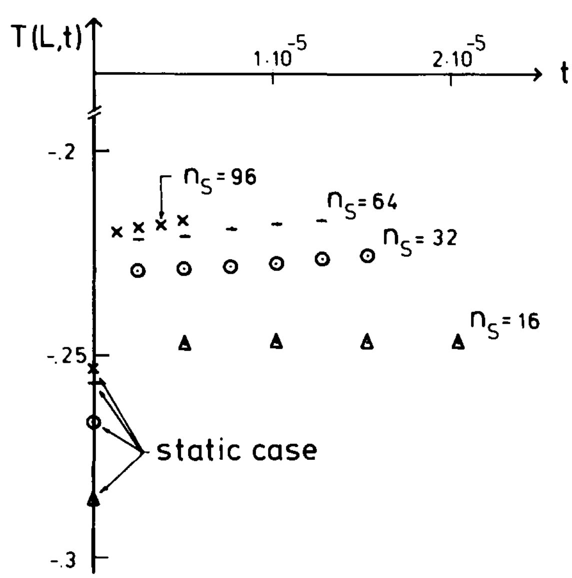

Figure A.1. Normal force $T(L, t)$ at tip, both static and dynamic, for different values of n_s.

where $\bar{W}$ and $\bar{V}$ are defined by (3.25), (3.26) and (3.27). From (A.1) it is seen that $\bar{m}_t = \bar{m}_a = .0125$. If we extrapolate the dynamic normal force $T(L, t)$ with $t > 0$ to time zero, we find a value unequal to the static normal force $T_1(L)$ at the tip. This static force is indicated at the vertical axis of Figure A.1. The magnitude of the jump appeared to be dependent on the mass $\bar{m}_t$ at the tip. It is zero for $\bar{m}_t = 0$. For increasing values of $\bar{m}_t$ it first increases but then decreases, such that for $\bar{m}_t \to \infty$ the jump tends to zero again.

This jump phenomenon seems to be related to the inextensibility of the bow by which possibly longitudional disturbances can be transferred instantaneously. In order to investigate this we replaced the constraint (2.31), expressing the inextensibility of the bow by the relation

$$T(s, t) = \tfrac{1}{2}\, \gamma\, U(s) \{x'^2 + y'^2 - 1\}, \tag{A.2}$$

where $U(s)$ is the distribution function of the strain stiffness (cross-sectional area times Young's modulus) of the bow and γ is a parameter.

Increasing values of γ correspond to less extensibility of the bow. In Figure A.2 the normal force is shown as a function of time, again immediately after the release of the arrow. The stiffness parameter γ ranges through the values 1, 10, 100, 10000. Also the curve for an inextensible bow ($\gamma \to \infty$) is drawn. It can be seen that, if the bow is definitely extensible, $\gamma = 1$, 10 or 100, the normal force at the tip is continuous with respect to time at $t = 0$. If the strain stiffness is increased the obtained curve 'converges' to the curve in the inextensible case and a jump appears. For values of s, $L_0 \leq s \leq L$, we observed the same behaviour of the normal force.

We mention that for a consistent treatment of an extensible bow the Euler-Bernoulli equation (2.32) has to be changed also, because then the parameter s is no longer the length parameter. However, by the foregoing results it is at least reasonable that the inextensibility of the bow has a strong influence on the behaviour of T after the release of the arrow.

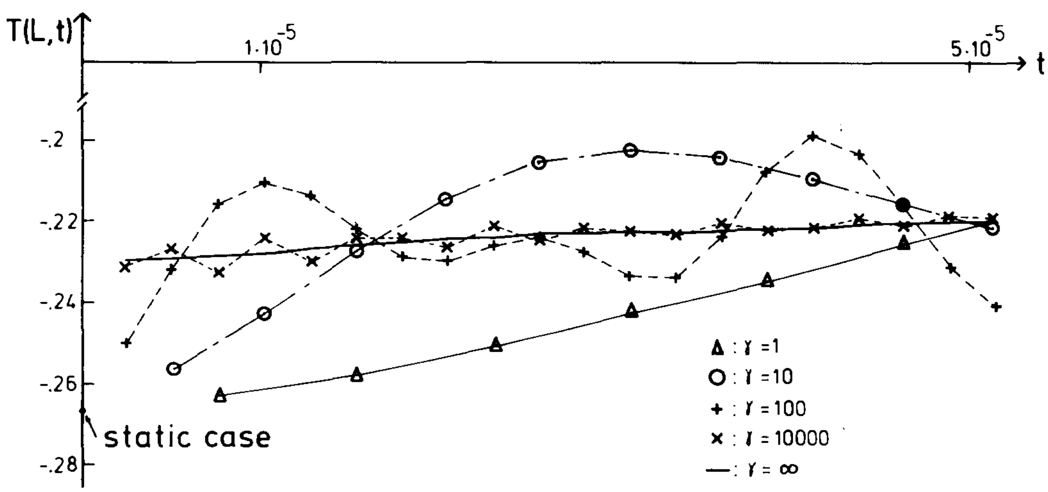

Figure A.2. Normal force $T(L, t)$ at tip for different values of γ.

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